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A Note on Time-Sharing

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ABSTRACT

The setting for the problem discussed here is a service facility which is to be "time-shared" by two customers. A precise notion of a processing schedule, which prescribes the times at which the facility is available to each customer, is introduced. Associated with each schedule is the expected total waiting time of the two customers. The schedules which minimize this time are called optimum schedules and are determined here. A number of examples and extensions are given which indicate the scope of the methods used.

1. STATEMENT OF THE PROBLEM

In this paper we shall study a problem connected with the time-sharing of a service facility by two customers. A processing schedule is introduced to prescribe the times at which the facility is available to each of the customers. The schedule is chosen so as to minimize the total delay which this time-sharing causes. The optimum schedules, in a sense to be made precise shortly, will be found.

Two customers, whom we shall denote by "0" and "1," have jobs which require $T^{(0)}$ and $T^{(1)}$ units of service time, respectively. We assume that the $\{T^{(j)}\}$ are independent random variables on some probability space $(\Omega, \mathscr{E}, \Pr)$. We further suppose that the values assumed by the $\{T^{(j)}\}$ are positive integers and that there exist integers $N^{(0)}$ and $N^{(1)}$ such that $\Pr\{\omega\colon T^{(j)}(\omega) > N^{(j)}\} = 0 \quad (j = 0, 1)$. *A processing schedule* $\mathbf{x}$ is a sequence $\mathbf{x} = (x_1, x_2, \ldots, x_N)$ which satisfies the conditions:

(i) $$N = N^{(0)} + N^{(1)},$$

(ii) $$x_i \in \{0, 1\}\ (1 \leqslant i \leqslant N),$$

(iii) $$x_1 + x_2 + \cdots + x_N = N^{(1)}.$$

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Define[1]

$$\tau_{\mathbf{x}}^{(1)}(\omega) = 1 + \sum_{s=1}^{N} \chi_{(0,N]} \Big(T^{(1)}(\omega) - \sum_{j=1}^{s} x_j\Big),$$

$$\tau_{\mathbf{x}}^{(0)}(\omega) = 1 + \sum_{s=1}^{N} \chi_{(0,N]} \Big(T^{(0)}(\omega) - \sum_{j=1}^{s} (1 - x_j)\Big),$$

$$\tau_{\mathbf{x}}(\omega) = \min(\tau_{\mathbf{x}}^{(0)}(\omega), \tau_{\mathbf{x}}^{(1)}(\omega)).$$

Note that

$$\tau_{\mathbf{x}}(\omega) = 1 + \sum_{s=1}^{N} \Big\{\chi_{(0,N]} \Big(T^{(0)}(\omega) - \sum_{j=1}^{s} (1 - x_j)\Big) \chi_{(0N,]} \Big(T^{(1)}(\omega) - \sum_{j=1}^{s} x_j\Big)\Big\}$$

The processing schedule $\mathbf{x}$ is to be interpreted as follows: in the time interval $i - 1 < t \leqslant i$ with $i \leqslant \tau_{\mathbf{x}}(\omega)$ the service facility is to work on the job of the customer whose name is "x_i." At time $\tau_{\mathbf{x}}(\omega)$ the servicing of one of the two customers will be completed; thereafter the service facility is devoted to finishing the job of the remaining customer.

We associate with the processing schedule $\mathbf{x}$ the random variables $\{D_{\mathbf{x}}^{(j)}\}$ which measure the delays the time-sharing causes. More precisely

$$D_{\mathbf{x}}^{(j)}(\omega) = \begin{cases} \tau_{\mathbf{x}}^{(j)}(\omega) - T^{(j)}(\omega), & \text{if } \tau_{\mathbf{x}}(\omega) = \tau_{\mathbf{x}}^{(j)}(\omega), \\ T^{(1-j)}(\omega) & \text{otherwise.} \end{cases}$$

The total delay to "0" *and* "1" is defined by $D_{\mathbf{x}} = D_{\mathbf{x}}^{(0)} + D_{\mathbf{x}}^{(1)}$ and *the average total to* "0" *and* "1" by

$$d(\mathbf{x}) = E(D_{\mathbf{x}}) = \int_{\Omega} D_{\mathbf{x}}(\omega) \Pr(d\omega).$$

We note that $D_{\mathbf{x}} = \tau_{\mathbf{x}}$. Our problem is to characterize the processing schedules $\mathbf{x}^*$ which minimize $d(\mathbf{x})$.

2. Calculation of $d(\mathbf{x})$

Let

$$p^{(j)}(i) = \Pr\{\omega: T^{(j)}(\omega) = i\}$$

and

$$Q^{(j)}(i) = \Pr\{\omega: T^{(j)}(\omega) > i\}.$$

[1] χ_A denotes the characteristic function of the set A.

It will be convenient to introduce an alternate notation for the processing schedules. Throughout this paper $s_1, s_2, \ldots, s_m, t_1, t_2, \ldots, t_m$ will denote integers which satisfy

$$0 \leqslant s_1 < s_2 < \cdots < s_{m-1} < s_m = N^{(0)},$$
$$0 < t_1 < t_2 < \cdots < t_{m-1} \leqslant t_m = N^{(1)}.$$

We use the notation

$$((0)_{s_1}, (1)_{t_1}, (0)_{s_2-s_1}, (1)_{t_2-t_1}, \ldots, (0)_{s_m-s_{m-1}}, (1)_{t_m-t_{m-1}})$$

to denote the processing schedule

$$(\underbrace{0, 0, \ldots, 0}_{\substack{s_1 \\ \text{0's}}}, \underbrace{1, 1, \ldots, 1}_{\substack{t_1 \\ \text{1's}}}, \underbrace{0, 0, \ldots, 0}_{\substack{s_2 - s_1 \\ \text{0's}}}, \underbrace{1, 1, \ldots, 1}_{\substack{t_2 - t_1 \\ \text{1's}}}, \ldots \underbrace{0, 0, \ldots, 0}_{\substack{s_m - s_{m-1} \\ \text{0's}}}, \underbrace{1, 1, \ldots, 1}_{\substack{t_m - t_{m-1} \\ \text{1's}}}).$$

Let

$$S_{\mathbf{x}}(k) = \begin{cases} 0, & \text{if } k = 0, \\ x_1 + x_2 + \cdots + x_k, & \text{if } 1 \leqslant k \leqslant N. \end{cases}$$

LEMMA 1. *If*

$$\mathbf{x} = ((0)_{s_1}, (1)_{t_1}, (0)_{s_2-s_1}, (1)_{t_2-t_1}, \ldots, (0)_{s_m-s_{m-1}}, (1)_{t_m-t_{m-1}}),$$

then

(i)

$$d(\mathbf{x}) = \sum_{k=0}^{N} Q^{(0)}(k - S_{\mathbf{x}}(k))\, Q^{(1)}(S_{\mathbf{x}}(k)), \tag{2.1}$$

(ii)

$$d(\mathbf{x}) = \sum_{i=1}^{m} \left\{ Q^{(1)}(t_{i-1}) \sum_{k=s_{i-1}}^{s_i-1} Q^{(0)}(k) + Q^{(0)}(s_i) \sum_{k=t_{i-1}}^{t_i-1} Q^{(1)}(k) \right\}.$$

PROOF:

$$d(\mathbf{x}) = E(\tau_{\mathbf{x}}) = 1 + \sum_{s=1}^{N} E\{\chi_{(0,N]}(T^{(0)} - (k - S_{\mathbf{x}}(k))\, \chi_{(0,N]}(T^{(1)} - S_{\mathbf{x}}(k))\},$$

and thus (i) is a consequence of the indepedence of the $\{T^{(j)}\}$. To prove (ii) we merely note that (i) may be written in the form

$$d(\mathbf{x}) = \sum_{i=1}^{m} \left\{ \sum_{k=t_{i-1}+s_{i-1}}^{t_{i-1}+s_i-1} Q^{(0)}(k - S_{\mathbf{x}}(k))\, Q^{(1)}(S_{\mathbf{x}}(k)) \right.$$

$$\left. + \sum_{k=t_{i-1}+s_i}^{t_i+s_i-1} Q^{(0)}(k - S_{\mathbf{x}}(k))\, Q^{(1)}(S_{\mathbf{x}}(k)) \right\}$$

Equation (2.1) may be viewed as a discrete "line integral." Consider the rectangle $\{(x, y)\colon 0 \leqslant x \leqslant N^{(0)}, 0 \leqslant y \leqslant N^{(1)}\}$; by *a path from* (0, 0) *to* $(N^{(0)}, N^{(1)})$ we mean a function $\mathbf{P} : \mathbf{p} = (p_0, p_1)$ with domain $\{0, 1, 2,..., (N^{(0)} + N^{(1)})\}$ and range in R^2 which satisfies the conditions:

(i) $$\mathbf{p}(0) = \mathbf{0},$$

(ii) $$\mathbf{p}(N^{(0)} + N^{(1)}) = (N^{(0)}, N^{(1)}),$$

(iii) $$\mathbf{p}(n) - \mathbf{p}(n-1) \in \{(0, 1), (1, 0)\} \; (1 \leqslant n \leqslant N^{(0)} + N^{(1)}).$$

The "line integral" $\int_P$ is defined by

$$\int_P = \sum_{n=0}^{N} Q^{(0)}(p_0(n))\, Q^{(1)}(p_1(n)).$$

If we associate with the processing schedule

$$\mathbf{x} = ((0)_{s_1}, (1)_{t_1}, (0)_{s_2-s_1}, (1)_{t_2-t_1}, ..., (0)_{s_m-s_{m-1}}, (1)_{t_m-t_{m-1}})$$

the path

$$\mathbf{p}(n) = (n - S_{\mathbf{x}}(n), S_{\mathbf{x}}(n)),$$

then $d(\mathbf{x}) = \int_P$. Our immediate objective is to employ Stoke's theorem and replace the "line integral" by an "area integral."

Define

$$m(i, j) = p^{(0)}(i + 1)\, Q^{(1)}(j) - Q^{(0)}(i)\, p^{(1)}(j + 1)$$

and

$$H(a, b; c, d) = \sum_{a \leqslant i < b} \sum_{c \leqslant j < d} m(i, j).$$

LEMMA 2.

$$d(((0)_{s_1}, (1)_{t_1}, \ldots, (0)_{s_m-s_{m-1}}, (1)_{t_m-t_{m-1}}))$$

$$= d(((0)_{N^{(0)}}, (1)_{N^{(1)}})) + \sum_{i=1}^{m-1} H(s_i, s_{i+1}; 0, t_i) \tag{2.2}$$

$$= d(((1)_{N^{(1)}}, (0)_{N^{(0)}})) - \sum_{i=0}^{m-1} H(0, s_{i+1}; t_i, t_{i+1}). \tag{2.3}$$

PROOF: We begin by noting that

$$m(i,j) = \{Q^{(0)}(i)\, Q^{(1)}(j) + Q^{(0)}(i)\, Q^{(1)}(j+1) + Q^{(0)}(i+1)\, Q^{(1)}(j+1)\}$$
$$- \{Q^{(0)}(i)\, Q^{(1)}(j) + Q^{(0)}(i+1)\, Q^{(1)}(j) + Q^{(0)}(i+1)\, Q^{(1)}(j+1)\}.$$

Therefore

$$\sum_{c \leqslant j < d} m(i,j) = \Big\{Q^{(0)}(i) \sum_{c \leqslant k < d} Q^{(1)}(k) + Q^{(0)}(i)\, Q^{(1)}(d)\Big\}$$
$$- \Big\{Q^{(0)}(i+1) \sum_{c \leqslant k < d} Q^{(1)}(k) + Q^{(0)}(i)\, Q^{(1)}(c)\Big\}. \tag{2.4}$$

Summing (2.4) over $a \leqslant i < b$ yields

$$H(a, b; c, d) = \Big\{Q^{(0)}(a) \sum_{c \leqslant k < d} Q^{(1)}(k) + Q^{(1)}(d) \sum_{a \leqslant k < b} Q^{(0)}(k)\Big\}$$
$$- \Big\{Q^{(0)}(b) \sum_{c \leqslant k < d} Q^{(1)}(k) + Q^{(1)}(c) \sum_{a \leqslant k < b} Q^{(0)}(k)\Big\}.$$

Finally

$$\sum_{i=1}^{m-1} H(s_i, s_{i+1}; 0, t_i) = \sum_{i=1}^{m-1} \Big\{Q^{(0)}(s_i) \sum_{k=0}^{t_i-1} Q^{(1)}(k) + Q^{(1)}(t_i) \sum_{k=s_i}^{s_{i+1}-1} Q^{(0)}(k)$$
$$- Q^{(0)}(s_{i+1}) \sum_{k=0}^{t_i-1} Q^{(1)}(k) - Q^{(1)}(0) \sum_{k=s_i}^{s_{i+1}-1} Q^{(0)}(k)\Big\}$$

which upon rearrangement yields the formula of Lemma 1(ii) for

$$d(((0)_{s_1}, (1)_{t_1}, \ldots, (0)_{s_m-s_{m-1}}, (1)_{t_m-t_{m-1}})) - d(((0)_{N^{(0)}}, (1)_{N^{(1)}})).$$

The proof of (2.3) depends upon the observation that

$$\sum_{i=0}^{N^{(0)}-1} \sum_{j=0}^{N^{(1)}-1} m(i,j) = d(((1)_{N^{(1)}}, (0)_{N^{(0)}})) - d(((0)_{N^{(0)}}, (1)_{N^{(1)}}))$$

$$= \sum_{i=1}^{m-1} H(s_i, s_{i+1}; 0, t_i) + \sum_{i=0}^{m-1} H(s_i, s_{i+1}; t_i, N^{(1)})$$

$$= \sum_{i=1}^{m-1} H(s_i, s_{i+1}; 0, t_i) + \sum_{i=0}^{m-1} H(0, s_{i+1}, t_i, t_{i+1}).$$

We note in passing that

$$E(T^{(0)}) = d(((0)_{N^{(0)}}, (1)_{N^{(1)}})), \qquad E(T^{(1)}) = d(((1)_{N^{(1)}}, (0)_{N^{(0)}})).$$

LEMMA 3. *If* $0 \leqslant a_0 \leqslant a_1 \leqslant a_2 \leqslant N^{(0)}$ *and* $0 \leqslant b_0 \leqslant b_1 \leqslant b_2 \leqslant N^{(1)}$ *then*

$$\begin{aligned} &H(a_1, a_2; b_0, b_1) \sum_{\substack{a_0 \leqslant k < a_1 \\ b_1 \leqslant h < b_2}} Q^{(0)}(k)\, Q^{(1)}(h) \\ &\quad + H(a_0, a_1; b_1, b_2) \sum_{\substack{a_1 \leqslant i < a_2 \\ b_0 \leqslant j < b_1}} Q^{(0)}(i)\, Q^{(1)}(j) \\ &= H(a_1, a_2; b_1, b_2) \sum_{\substack{a_0 \leqslant k < a_1 \\ b_0 \leqslant j < b_1}} Q^{(0)}(k)\, Q^{(1)}(j) \\ &\quad + H(a_0, a_1; b_0, b_1) \sum_{\substack{a_1 \leqslant i < a_2 \\ b_1 \leqslant h < b_2}} Q^{(0)}(i)\, Q^{(1)}(h). \end{aligned} \tag{2.5}$$

In particular (2.5) *implies that*

$$\{\operatorname{sgn} H(a_1, a_2; b_0, b_1) + \operatorname{sgn} H(a_0, a_1; b_1 b_2)\}$$
$$\{\operatorname{sgn} H(a_0, a_1; b_0, b_1) + \operatorname{sgn} H(a_1, a_2; b_1, b_2)\} \geqslant 0, \tag{2.6}$$

where

$$\operatorname{sgn} x = \begin{cases} 1, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ -1, & \text{if } x < 0. \end{cases}$$

PROOF: The proof of (2.5) is by direct computation and we omit the details.

REMARK. We note the following generalization of (2.5). If

$$0 \leqslant a_0 \leqslant a_1 \leqslant a_2 \leqslant a_3 \leqslant N^{(0)},$$
$$0 \leqslant b_0 \leqslant b_1 \leqslant b_2 \leqslant b_3 \leqslant N^{(1)},$$

then

$$\begin{aligned} & H(a_0, a_1; b_0, b_1) \sum_{\substack{a_2 \leqslant k < a_3 \\ b_2 \leqslant h < b_3}} Q^{(0)}(k)\, Q^{(1)}(h) \\ & \quad + H(a_2, a_3; b_2, b_3) \sum_{\substack{a_0 \leqslant i < a_1 \\ b_0 \leqslant h < b_1}} Q^{(0)}(i)\, Q^{(1)}(h) \\ & = H(a_2, a_3; b_0, b_1) \sum_{\substack{a_0 \leqslant k < a_1 \\ b_2 \leqslant h < b_3}} Q^{(0)}(k)\, Q^{(1)}(h) \\ & \quad + H(a_0, a_1; b_2, b_3) \sum_{\substack{a_2 \leqslant i < a_3 \\ b_0 \leqslant j < b_1}} Q^{(0)}(i)\, Q^{(1)}(j). \end{aligned} \tag{2.7}$$

3. OPTIMALITY CRITERIA FOR $((0)_{s_1}, (1)_{t_1}, \ldots, (0)_{s_m - s_{m-1}}, (1)_{t_m - t_{m-1}})$

LEMMA 4. *If* $((0)_{s_1}, (1)_{t_1}, \ldots, (0)_{s_m - s_{m-1}}, (1)_{t_m - t_{m-1}})$ *is an optimum processing schedule then*

(i) $H(s_i, \alpha; \beta, t_i) \leqslant 0,$

$$\textit{for } s_i \leqslant \alpha \leqslant s_{i+1},\ t_{i-1} \leqslant \beta \leqslant t_i,\ \textit{and } 1 \leqslant i < m, \tag{3.1}$$

(ii) $H(\alpha, s_{i+1}; t_i, \beta) \geqslant 0,$

$$\textit{for } s_i \leqslant \alpha \leqslant s_{i+1},\ t_i \leqslant \beta \leqslant t_{i+1},\ \textit{and } 0 \leqslant i < m. \tag{3.2}$$

PROOF: Suppose on the contrary that $H(s_{i_0}, \alpha_0; \beta_0, t_{i_0}) > 0$; we define a processing schedule $s_1', s_2', \ldots, s_{m+1}', t_1', t_2', \ldots, t_{m+1}'$ by

$$s_i' = \begin{cases} s_i, & \text{if } 1 \leqslant i \leqslant i_0, \\ \alpha_0, & \text{if } i = i_0 + 1, \\ s_{i-1}, & \text{if } i_0 + 1 < i \leqslant m + 1, \end{cases}$$

$$t_i' = \begin{cases} t_i, & \text{if } 1 \leqslant i < i_0, \\ \beta_0, & \text{if } i = i_0, \\ t_{i-1}, & \text{if } i_0 < i \leqslant m + 1. \end{cases}$$

It is easily seen that

$$d(((0)_{s_1'}, (1)_{t_1'}, \ldots, (0)_{s_{m+1}'-s_m'}, (1)_{t_{m+1}'-t_m'}))$$
$$= d(((0)_{s_1}, (1)_{t_1}, \ldots, (0)_{s_m-s_{m-1}}, (1)_{t_m-t_{m-1}})) - H(s_{i_0}, \alpha_0; \beta_0, t_{i_0}),$$

which is a contradiction to the assumed optimality of

$$((0)_{s_1}, (1)_{t_1}, \ldots, (0)_{s_m-s_{m-1}}, (1)_{t_m-t_{m-1}}).$$ [2]

The proof of Lemma 4(ii) is similar and we omit the details.

LEMMA 5. *If* $((0)_{s_1}, (1)_{t_1}, \ldots, (0)_{s_m-s_{m-1}}, (1)_{t_m-t_{m-1}})$ *is an optimum schedule then*

(i) $H(s_i, s_{i+1}; \beta, t_{i-1}) \leqslant 0,$ *for* $0 \leqslant \beta \leqslant t_{i-1}$ *and* $1 < i \leqslant m,$ (3.3)

(ii) $H(\alpha, s_i; t_i, t_{i+1}) \geqslant 0,$ *for* $0 \leqslant \alpha \leqslant s_i$ *and* $1 \leqslant i < m.$ (3.4)

PROOF: We note that

$H(s_1, s_2; \beta, t_1) \leqslant 0$ (by (3.1) with $i = 1$, $\alpha = s_2$, and $0 \leqslant \beta \leqslant t_1$),
$H(s_1, s_2; t_1, t_2) \geqslant 0$ (by (3.2) with $i = 1$, $\alpha = s_1$, and $\beta = t_2$),
$H(s_2, s_3; t_1, t_2) \leqslant 0$ (by (3.1) with $i = 2$, $\alpha = s_3$, and $\beta = t_1$).

Thus by (2.6) applied to the rectangles determined by the system of points[3]

$$\begin{array}{lll} a_0 = s_1, & a_1 = s_2, & a_2 = s_3, \\ b_0 = \beta, & b_1 = t_1, & b_2 = t_2, \end{array} \tag{3.5}$$

we are able to deduce (3.3) for $i = 2$. The proof of (3.3) is by induction; let us suppose therefore that

(#) $H(s_i, s_{i+1}; \beta, t_{i-1}) \leqslant 0$ for $0 \leqslant \beta \leqslant t_{i-1}$ and $1 < i < q$.

Then by (#) with $i = q - 1$ and (3.1) (with $i = q - 1$, $\alpha = s_q$, $\beta = t_{q-2}$) we have

$$H(s_{q-1}, s_q; \beta, t_{q-1}) = H(s_{q-1}, s_q; t_{q-2}, t_{q-1}) + H(s_{q-1}, s_q; \beta, t_{q-2}) \leqslant 0$$
$$(0 \leqslant \beta \leqslant t_{q-2}).$$

[2] We make the obvious interpretation of $((0)_{s_1'}, (1)_{t_1'}, \ldots, (0)_{s_{m+1}'-s_m'}, (1)_{t_{m+1}'-t_m'})$, if $t_{i+1}' = t_i'$ or $s_{i+1}' = s_i'$.

[3] Henceforth we shall employ the notation $(a_0, a_1, a_2; b_0, b_1, b_2)$ for the display of equation (3.5).

Next we observe that

$$H(s_q, s_{q+1}; t_{q-1}, t_q) \leqslant 0 \text{ (by (3.1) with } i = q, \alpha = s_{q+1}, \beta = t_{q-1}),$$
$$H(s_{q-1}, s_q; t_{q-1}, t_q) \geqslant 0 \text{ (by (3.2) with } i = q-1, \alpha = s_{q-1}, \beta = t_q).$$

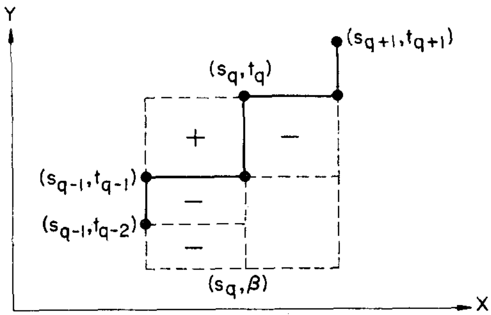

FIGURE 1

These regions of positive and negative "area" are displayed in Figure 1 Applying (2.6) to the rectangles determined by the system of points $(s_{q-1}, s_q, s_{q+1}; \beta, t_{q-1}, t_q)$ we obtain (#) for $i = q$. This proves Lemma 5(i). The proof of Lemma 5(ii) is similar and we omit the details.

LEMMA 6. *If* $((0)_{s_1}, (1)_{t_1}, ..., (0)_{s_m - s_{m-1}}, (1)_{t_m - t_{m-1}})$ *is an optimum schedule then*

(i) $H(s_i, \gamma; \delta, t_{i-1}) \leqslant 0$

$$\text{for } s_i \leqslant \gamma \leqslant N^{(0)}, 0 \leqslant \delta \leqslant t_{i-1} \text{ and } 1 < i < m, \tag{3.6}$$

(ii) $H(\gamma, s_{i+1}; t_{i+1}, \delta) \geqslant 0$

$$\text{for } 0 \leqslant \gamma \leqslant s_{i+1}, t_{i+1} \leqslant \delta \leqslant N^{(1)}, \text{ and } 0 \leqslant i < m. \tag{3.7}$$

PROOF: We begin by noting that

$$H(s_{m-1}, \alpha; t_{m-2}, t_{m-1}) \leqslant 0 \text{ (by (3.1) with } i = m-1, \beta = t_{m-2}, \text{ and } s_{m-1} \leqslant \alpha \leqslant s_m),$$

$$H(s_{m-2}, s_{m-1}; t_{m-2}, t_{m-1}) \geqslant 0 \text{ (by (3.2) with } i = m-2, \alpha = s_{m-1}, \text{ and } \beta = t_{m-1}),$$

$$H(s_{m-2}, s_{m-1}; \beta, t_{m-2}) \leqslant 0 \text{ (by (3.3) with } i = m-2, \alpha = s_{m-1}, \text{ and } 0 \leqslant \beta \leqslant t_{m-2})$$

Thus by applying (2.6) to the rectangles determined by the system of points $(s_{m-2}, s_{m-1}, \alpha; \beta, t_{m-2}, t_{m-1})$ we obtain (3.6) for $i = m - 1$, The proof of (3.6) is by induction; let us suppose

$$(\#) \qquad H(s_{m-p}, \gamma; \delta, t_{m-p-1}) \leqslant 0 \text{ for } s_{m-p} \leqslant \gamma \leqslant N^{(0)}, 0 \leqslant \delta \leqslant t_{m-p-1} \text{ and } 1 \leqslant p < p_0 .$$

CASE I. $s_{m-p_0+1} \leqslant \gamma \leqslant N^{(0)}$.

By (#) with $p = p_0 - 1, \delta = t_{m-p_0-1}$, and (3.1) (with $i = m - p_0$, $\alpha = s_{m-p_0+1}, \beta = t_{m-p_0-1}$) we have

$$H(s_{m-p_0}, \gamma; t_{m-p_0-1}, t_{m-p_0}) = H(s_{m-p_0}, s_{m-p_0+1}; t_{m-p_0-1}, t_{m-p_0}) + H(s_{m-p_0+1}, \gamma; t_{m-p_0-1}, t_{m-p_0}) \leqslant 0. \tag{3.8}$$

Next

$$H(s_{m-p_0-1}, s_{m-p_0}; t_{m-p_0-1}, t_{m-p_0}) \geqslant 0 \tag{3.9}$$

(by (3.2) with $i = m - p_0 - 1$, $\alpha = s_{m-p_0-1}$, and $\beta = t_{m-p_0}$).

We assert that

$$H(s_{m-p_0-1}, s_{m-p_0}; \delta, t_{m-p_0-1}) \leqslant 0 \quad \text{for} \quad 0 \leqslant \delta \leqslant t_{m-p_0-1}. \tag{3.10}$$

If $t_{m-p_0-2} \leqslant \delta \leqslant t_{m-p_0-1}$, then (3.10) follows from (3.1) (with $i = m - p_{0-1}$, $\alpha = s_{m-p_0}$, and $\beta = \delta$). If $\delta < t_{m-p_0-2}$ then we write

$$H(s_{m-p_0-1}, s_{m-p_0}; \delta, t_{m-p_0-1}) = H(s_{m-p_0-1}, s_{m-p_0}; \delta, t_{m-p_0-2}) + H(s_{m-p_0-1}, s_{m-p_0}; t_{m-p_0-2}, t_{m-p_0-1}). \tag{3.11}$$

The second term (in the right-hand side member) of (3.11) is non-positive by (3.1) (with $i = m - p_0 - 1$, $\alpha = s_{m-p_0}$, $\beta = t_{m-p_0-2}$) and the first term (in the right-hand side member) of (3.11) is likewise non-positive by Lemma 5(i) (with $i = m - p_0 - 1$ and $\beta - \delta$). Thus (3.10) is established.

Finally using (3.8)–(3.10) we apply (2.6) to the rectangles determined by the system of points $(s_{m-p_0-1}, s_{m-p_0}, \gamma; \delta, t_{m-p_0-1}, t_{m-p_0})$ to obtain (#) for $p = p_0$ and $s_{m-p_0+1} \leqslant \gamma \leqslant N^{(0)}$. The regions of positive and negative area are shown in Figure 2.

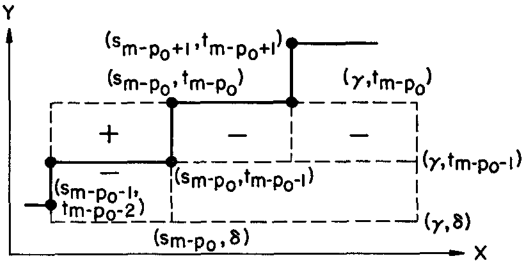

FIGURE 2

CASE II. $s_{m-p_0} \leqslant \gamma \leqslant s_{m-p_0+1}$.

We have

$$H(s_{m-p_0}, \gamma; t_{m-p_0-1}, t_{m-p_0}) \leqslant 0$$

$$\text{(by (3.1) with } i = m - p_0, \quad \alpha = \gamma, \quad \text{and} \quad \beta = t_{m-p_0-1}). \tag{3.12}$$

Using (3.8), (3.9), and (3.12) we apply (2.6) to the rectangles determined by the system of points $(s_{m-p_0-1}, s_{m-p_0}, \gamma; \delta, t_{m-p_0-1}, t_{m-p_0})$ to prove (#) for $p = p_0$ and $s_{m-p_0} \leqslant \gamma \leqslant s_{m-p_0+1}$. The regions of positive and negative area for Case II are shown in Figure 3. This completes the proof of (#) for $p = p_0$ and therefore the proof of Lemma 6(i).

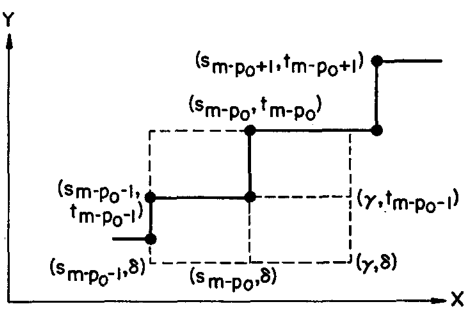

FIGURE 3

The proof of Lemma 6(ii) is similar and we omit the details.

LEMMA 7. *Let* $((0)_{s_1}, (1)_{t_1}, \ldots, (0)_{s_m - s_{m-1}}, (1)_{t_m - t_{m-1}})$ *be an optimum schedule. Suppose* i_0 *is such that*

$$H(s_i, N^{(0)}; 0, t_i) = 0, \qquad 1 \leqslant i < i_0,$$
$$H(s_{i_0}, N^{(0)}; 0, t_{i_0}) \leqslant 0, \qquad 1 < i_0 < m.$$

Then for the schedule defined by

$$s_1', s_2', \ldots, s_{m-i_0+1}', \qquad t_1', t_2', \ldots, t_{m-i_0+1}',$$

$$s_i' = s_{i_0+i-1}, \qquad 1 \leqslant i \leqslant m - i_0 + 1,$$

$$t_i' = t_{i_0+i-1}, \qquad 1 \leqslant i \leqslant m - i_0 + 1,$$

we have

$$\begin{aligned} &d(((0)_{s_1}, (1)_{t_1}, \ldots, (0)_{s_m - s_{m-1}}, (1)_{t_m - t_{m-1}})) \\ &\quad = d(((0)_{s_1'}, (1)_{t_1'}, \ldots, (0)_{s'_{m-i_0+1} - s'_{m-i_0}}, (1)_{t'_{m-i_0+1} - t'_{m-i_0}})) \end{aligned}$$

PROOF: We have

$$\begin{aligned} &d(((0)_{s_1}, (1)_{t_1}, \ldots, (0)_{s_m - s_{m-1}}, (1)_{t_m - t_{m-1}}) \\ &\quad = d(((0)_{s_1'}, (1)_{t_1'}, \ldots, (0)_{s'_{m-i_0+1} - s_{m-i_0}}, (1)_{t'_{m-i_0+1} - t_{m-i_0}})) \\ &\quad\quad + \sum_{i=1}^{i_0-1} H(s_i, s_{i+1}; 0, t_i). \end{aligned}$$

Now

$$\begin{aligned} H(s_i, N^{(0)}; 0, t_i) = H(s_i, s_{i+1}; 0, t_{i-1}) &+ H(s_i, s_{i+1}; t_{i-1}, t_i) \\ &+ H(s_{i+1}, N^{(0)}; 0, t_i), \end{aligned} \tag{3.13}$$

and by Lemmas 4(i) and 6(i) each of the terms on the right-hand side of (3.13) are non-positive. On the other hand for $1 \leqslant i < i_0$ we have $H(s_i, N^{(0)}; 0, t_i) = 0$ and therefore we may conclude that

$$H(s_i, s_{i+1}; 0, t_{i-1}) = H(s_i, s_{i+1}; t_{i-1}, t_i) = H(s_{i+1}, N^{(0)}; 0, t_i) = 0, \quad 1 \leqslant i < i_0.$$

Since

$$H(s_i, s_{i+1}; 0, t_i) = H(s_i, s_{i+1}; 0, t_{i-1}) + H(s_i, s_{i+1}; t_{i-1}, t_i)$$

we have $H(s_i, s_{i+1}; 0, t_i) = 0 (1 \leqslant i < i_0)$ which proves the lemma.

LEMMA 8. *Let* $((0)_{s_1}, (1)_{t_1}, \ldots, (0)_{s_m - s_{m-1}}, (1)_{t_m - t_{m-1}})$ *be an optimum schedule. If* $H(s_1, N^{(0)}; 0, t_1) < 0$ *then*

$$H(s_i, N^{(0)}; 0, t_i) < 0 \qquad (1 \leqslant i < m).$$

PROOF: The proof is by induction; let us suppose

$$(\#) \qquad H(s_i, N^{(0)}; 0, t_i) < 0, \qquad 1 \leqslant i < i_0 .$$

Now

$$\begin{aligned} H(s_{i_0}, N^{(0)}; 0, t_{i_0}) = {} & H(s_{i_0}, N^{(0)}; 0, t_{i_0-1}) \\ & + H(s_{i_0}, s_{i_0+1}; t_{i_0-1}, t_{i_0}) \\ & + H(s_{i_0+1}, N^{(0)}; t_{i_0-1}, t_{i_0}) \end{aligned} \qquad (3.14)$$

Each of the terms appearing on the right-hand side of (3.14) is non-positive (by Lemmas 4 and 6). If we should assume that on the contrary $H(s_{i_0}, N^{(0)}; 0, t_{i_0}) = 0$ then each of these terms would be equal to zero. Since $H(s_{i_0-1}, N^{(0)}; 0, t_{i_0-1}) < 0$ we could then conclude

$$H(s_{i_0-1}, s_{i_0}; 0, t_{i_0-1}) < 0,$$

$$H(s_{i_0}, N^{(0)}; 0, t_{i_0-1}) = 0,$$

$$H(s_{i_0-1}, s_{i_0}; t_{i_0-1}, t_{i_0}) \geqslant 0,$$

$$H(s_{i_0}, N^{(0)}; t_{i_0-1}, t_{i_0}) \leqslant 0,$$

which yields a contradiction by applying (2.6) to the rectangles determined by the system of points $(s_{i_0-1}, s_{i_0}, N^{(0)}; 0, t_{i_0-1}, t_{i_0})$.

By employing the generalization of (2.5) (see the remark following the proof of Lemma 3) we can obtain

LEMMA 9. *If* $((0)_{s_1}, (1)_{t_1}, \ldots, (0)_{s_m-s_{m-1}}, (1)_{t_m-t_{m-1}})$ *is an optimum schedule then*

(i) $\qquad H(s_i, s_{i+1}; t_j, t_{j+1}) \leqslant 0, \qquad$ for $\quad 0 \leqslant j < i < m,$

(ii) $\qquad H(s_i, s_{i+1}; t_j, t_{j+1}) \geqslant 0, \qquad$ for $\quad 0 \leqslant i \leqslant j < m.$

4. DETERMINATION OF THE FIRST SWITCHING POINT

We shall indicate in this section how to determine s_1 if $s_1 > 0$, and t_1 if $s_1 = 0$. Successive switching points are then found by reapplying the argument to the "reduced" problem.

THEOREM 1. (i) *A necessary and sufficient condition that* $((0)_{N^{(0)}}, (1)_{N^{(1)}})$ *be an optimum schedule is*

$$\min_{\substack{0\leqslant a\leqslant N^{(0)}\\ 0\leqslant b\leqslant N^{(1)}}} H(a, N^{(0)}; 0, b) \geqslant 0. \tag{4.1}$$

(ii) *A necessary and sufficient condition that* $((1)_{N^{(1)}}, (0)_{N^{(0)}})$ *be an optimum schedule is*

$$\max_{\substack{0\leqslant a\leqslant N^{(0)}\\ 0\leqslant b\leqslant N^{(1)}}} H(0, a; b, N^{(1)}) \leqslant 0. \tag{4.2}$$

PROOF (necessity of (4.1)):
If on the contrary $H(a_0, N^{(0)}; 0, b_0) < 0$ then

$$\begin{aligned} &d(((0)_{a_0}, (1)_{b_0}, (0)_{N^{(0)}-a_0}, (1)_{N^{(1)}-b_0})) \\ &\quad = d(((0)_{N^{(0)}}, (1)_{N^{(1)}})) + H(a_0, N^{(0)}; 0, b_0) < d(((0)_{N^{(0)}}, (1)_{N^{(1)}})), \end{aligned}$$

which provides a contradiction.

PROOF (sufficiency of (4.1)):
Suppose that (4.1) holds but that $((0)_{N^{(0)}}, (1)_{N^{(1)}})$ is not an optimum schedule. If $((0)_{s_1}, (1)_{t_1}, \dots, (0)_{s_m-s_{m-1}}, (1)_{t_m-t_{m-1}})$ is an optimum schedule then we have

$$H(s_i, N^{(0)}; 0, t_i) \leqslant 0 \qquad (1 \leqslant i < m).$$

We assert that

$$\min_{1\leqslant i<m} H(s_i, N^{(0)}; 0, t_i) < 0. \tag{4.3}$$

For if $\min H(s_i, N^{(0)}; 0, t_i) = 0$ $(1 \leqslant i < m)$ then, by Lemma 7,

$$d(((0)_{s_1}, (1)_{t_1}, \dots, (0)_{s_m-s_{m-1}}, (1)_{t_m-t_{m-1}})) = d(((0)_{N^{(0)}}, (1)_{N^{(1)}})),$$

contrary to our assumption. Thus (4.3) holds and hence

$$\min_{\substack{0\leqslant a\leqslant N^{(0)}\\ 0\leqslant b\leqslant N^{(1)}}} H(a, N^{(0)}; 0, b) < 0,$$

contrary to hypothesis.
The proof of Theorem 1(ii) is similar and we omit the details.

Define

$$\mathcal{O}\!\ell^{(0)} = \{(i,j): 0 \leqslant i \leqslant N^{(0)}, 0 \leqslant j \leqslant N^{(1)}, H(i, N^{(0)}; 0, j) < 0\},$$

$$\mathcal{O}\!\ell^{(1)} = \{(i,j): 0 \leqslant i \leqslant N^{(0)}, 0 \leqslant j \leqslant N^{(1)}, H(0, i; j, N^{(1)}) > 0\},$$

$$\mathcal{S}^{(0)} = \begin{cases} \{(i,j) : (i,j) \in \mathcal{O}\!\ell^{(0)} \text{ and } \min\limits_{\substack{0\leqslant\alpha\leqslant i\\ 0\leqslant\beta\leqslant j}} H(\alpha, i; 0, \beta) \geqslant 0\}, & \text{if } \mathcal{O}\!\ell^{(0)} \neq \varnothing, \\ \{(N^{(0)}, N^{(1)}\}, & \text{if } \mathcal{O}\!\ell^{(0)} = \varnothing, \end{cases}$$

$$\mathcal{S}^{(1)} = \begin{cases} \{(i,j) : (i,j) \in \mathcal{O}\!\ell^{(1)} \text{ and } \max\limits_{\substack{0\leqslant\alpha\leqslant i\\ 0\leqslant\beta\leqslant j}} H(0, \alpha; \beta, j) \leqslant 0\}, & \text{if } \mathcal{O}\!\ell^{(1)} \neq \varnothing, \\ \{(N^{(0)}, N^{(1)})\}, & \text{if } \mathcal{O}\!\ell^{(1)} = \varnothing, \end{cases}$$

$$j_H = \max\{j: (i,j) \in \mathcal{S}^{(0)}\}, \; i_H = \max\{i: (i, j_H) \in \mathcal{S}^{(0)}\},$$

$$i_V = \max\{i: (i,j) \in \mathcal{S}^{(1)}\}, \; j_V = \max\{j: (i_V, j) \in \mathcal{S}^{(1)}\}.$$

LEMMA 10. *If* $\mathcal{O}\!\ell^{(i)} \neq \varnothing$ *then* $\mathcal{S}^{(i)} \neq \{(N^{(0)}, N^{(1)})\}$ $(i = 0, 1)$.

PROOF: Suppose $\mathcal{O}\!\ell^{(0)} \neq \varnothing$. Define $i^* = \min\{i: (i,j) \in \mathcal{O}\!\ell^{(0)}\}$ and $j^* = \min\{j: (i,j) \in \mathcal{O}\!\ell^{(0)}\}$. We assert that $(i^*, j^*) \in \mathcal{S}^{(0)}$. Suppose on the contrary that $H(a, i^*; 0, b) < 0$ for some pair (a, b) with $0 \leqslant a < i^*$ and $0 < b \leqslant j^*$. Note that

(a) $b < j^*$ (by the minimality of i^*),

(b) $H(i^*, N^{(0)}; b, j^*) < 0$ (by the minimality of j^*),

(c) $H(a, i^*; 0, j^*) > 0$ (by the minimality of i^*),

and hence $H(a, i^*; b, j^*) > 0$. Applying (2.6) to the rectangles determined by the system of points $(a, i^*, N^{(0)}; 0, b, j^*)$ we can conclude that $H(i^*, N^{(0)}; 0, b) < 0$ and hence

$$H(a, N^{(0)}; 0, b) = H(a, i^*; 0, b) + H(i^*, N^{(0)}; 0, b) < 0,$$

which contradicts the minimality of i^*. This proves $(i^*, j^*) \in \mathcal{S}^{(0)}$. Since $(N^{(0)}, N^{(1)}) \notin \mathcal{O}\!\ell^{(0)}$ we have $(i^*, j^*) \neq (N^{(0)}, N^{(1)})$. Let $i_H = i^*, j_V = j^*$.

THEOREM 2. *Let* $((0)_{s_1}, (1)_{t_1}, ..., (0)_{s_m - s_{m-1}}, (1)_{t_m - t_{m-1}})$ *be an optimum schedule. Suppose that among all optimum schedules it maximizes the value of* s_1. *Then* $s_1 = i_H$.

PROOF: If $H(s_i, N^{(0)}; 0, t_i) = 0 (1 \leqslant i < m)$ then by Lemma 7

$$d(((0)_{s_1}, (1)_{t_1}, ..., (0)_{s_m - s_{m-1}}, (1)_{t_m - t_{m-1}})) = d(((0)_{N^{(0)}}, (1)_{N^{(1)}})).$$

On the other hand the optimality of $((0)_{N^{(0)}}, (1)_{N^{(1)}})$ implies by Theorem 1 that $\mathcal{O}\!\mathcal{l}^{(0)} = \varnothing$ so that $i_H = N^{(0)} = s_1$ and $m = 1$.

Henceforth we assume that $\min H(s_i, N^{(0)}; 0, t_i) < 0$. By Lemma 7 and the maximality of s_1 we have $H(s_1, N^{(0)}; 0, t_1) < 0$. Next

$$H(a, s_1; 0, b) \geqslant 0 \text{ (by (3.2) with } i = 0, \alpha = a, \text{ and } \beta = b),$$

whenever $0 \leqslant a \leqslant s_1$ and $0 \leqslant b \leqslant t_1$. Thus $(s_1, t_1) \in \mathscr{S}^{(0)}$. Now we shall suppose on the contrary that $i_H > s_1$ and $j_H \geqslant t_1$. If $i_H \leqslant s_2$ then

$$H(s_1, i_H; 0, t_1) \leqslant 0 \text{ (by (3.1) with } i = 1, \alpha = i_H, \text{ and } \beta = 0),$$

while if $i_H > s_2$ then we write

$$H(s_1, i_H; 0, t_1) = H(s_1, s_2; 0, t_1) + H(s_2, i_H; 0, t_1),$$

where

$$H(s_1, s_2; 0, t_1) \leqslant 0 \text{ (by (3.1) with } i = 1, \alpha = s_2, \text{ and } \beta = 0),$$

$$H(s_2, i_H; 0, t_1) \leqslant 0 \text{ (by (3.6) with } i = 2, \gamma = i_H, \text{ and } \delta = 0).$$

In either case $H(s_1, i_H; 0, t_1) \leqslant 0$. By the maximality of s_1 we have $H(s_1, i_H; 0, t_1) < 0$ and this contradicts the fact that $(i_H, j_H) \in \mathscr{S}^{(0)}$. Thus $i_H \leqslant s_1$.

Now we suppose that $i_H < s_1$. By the definition of j_H we must have $j_H > t_1$. We shall deduce that $(s_1, j_H) \in \mathscr{S}^{(0)}$. This will contradict the maximality of i_H and thus prove the theorem.

If $\mu \leqslant t_2$ then

$$H(s_1, s_2; t_1, \mu) \geqslant 0 \text{ (by (3.2) with } i = 1, \alpha = s_1, \text{ and } \beta = \mu).$$

If $\mu > t_2$ then we write

$$H(s_1, s_2; t_1, \mu) = H(s_1, s_2; t_1, t_2) + H(s_1, s_2; t_2, \mu)$$

and note

$$H(s_1, s_2; t_1, t_2) \geqslant 0 \text{ (by (3.2) with } i = 1, \alpha = s_1, \text{ and } \beta = t_2),$$

$$H(s_1, s_2; t_2, \mu) \geqslant 0 \text{ (by (3.7) with } i = 1, \gamma = s_1, \text{ and } \delta = \mu),$$

so that for $\mu \geqslant t_1$ we have $H(s_1, s_2; t_1, \mu) \geqslant 0$. Furthermore

$$H(\alpha, s_1; 0, t_1) \geqslant 0 \text{ for } 0 \leqslant \alpha \leqslant s_1 \text{ (by (3.2) with } i = 0, \beta = t_1),$$

$$H(s_1, s_2; 0, t_1) < 0 \text{ (by (3.1) with } i = 1, \alpha = s_2, \beta = 0, \text{ and the maximality of } s_1).$$

Applying (2.6) to the rectangles determined by the system of points $(\alpha, s_1, s_2; 0, t_1, \mu)$ we obtain

$$H(\alpha, s_1; t_1, \mu) > 0, \qquad 0 \leqslant \alpha < s_1, \qquad t_1 < \mu,$$

and thus

$$H(\alpha, s_1; 0, \mu) > 0, \qquad 0 \leqslant \alpha < s_1, \qquad t_1 < \mu.$$

It now follows that

$$\min_{\substack{0 \leqslant \alpha \leqslant s_1 \\ 0 \leqslant \beta \leqslant j_H}} \neq H(\alpha, s_1; 0, \beta) \geqslant 0.$$

On the other hand the inequalities

$$H(i_H, s_1; 0, j_H) > 0, \qquad H(i_H, N^{(0)}; 0, j_H) < 0,$$

imply $H(s_1, N^{(0)}; 0, j_H) < 0$. This proves $(s_1, j_H) \in \mathscr{S}^{(0)}$.

A similar argument proves

THEOREM 3. *Let* $((0)_{s_1}, (1)_{t_1}, \ldots, (0)_{s_m - s_{m-1}}, (1)_{t_m - t_{m-1}})$ *be an optimum schedule. Suppose*

(a) $s_1 = 0$, *and*

(b) *among all optimum schedules with* $s_1 = 0$ *this schedule maximizes* t_1. *Then* $t_1 = j_V$.

5. THE SYMMETRIC CASE

We specialize in this section the results of Section 4 for the symmetric case, i.e., where $Q^{(0)} = Q^{(1)}$. Let us begin by noting that symmetry implies

$$H(a, b; a, b) = 0, \qquad 0 \leqslant a \leqslant b. \tag{5.1}$$

LEMMA 11. *Let*

$$0 \leqslant s_1 < s_2 < \cdots s_{m-1} < s_m = N,^4$$
$$0 < t_1 < t_2 < \cdots t_{m-1} \leqslant t_m = N$$

satisfy $s_i \geqslant t_i (1 \leqslant i < i_0)$ *and* $s_{i_0} < t_{i_0}$. *Then*

$$d(((0)_{s_1}, (1)_{t_1}, \ldots, (0)_{s_m - s_{m-1}}, (1)_{t_m - t_{m-1}}))$$
$$= d(((0)_{s_1'}, (1)_{t_1'}, \ldots, (0)_{s_{m+1}' - s_m'}, (1)_{t_{m+1}' - t_m'}))$$

[4] In the symmetric case we set $N = N^{(0)} = N^{(1)}$.

where

$$s_i' = \begin{cases} s_i\,, & 1 \leqslant i \leqslant i_0\,, \\ t_{i-1}\,, & i_0 < i \leqslant m+1, \end{cases}$$

$$t_i' = \begin{cases} t_i\,, & 1 \leqslant i \leqslant i_0\,, \\ s_i\,, & i_0 \leqslant i \leqslant m, \\ s_m\,, & i = m+1. \end{cases}$$

PROOF: By Lemma 1(ii)

$$d(((0)_{s_1'}\,,(1)_{t_1'}\,,\ldots,(0)_{s_{m+1}'-s_m'}\,,(1)_{t_{m+1}'-t_m'}))$$

$$= \sum_{i=1}^{m+1} \left\{ Q(t_{i-1}') \sum_{k=s_{i-1}'}^{s_1'-1} Q(k) + Q(s_i') \sum_{k=i_{i-1}'}^{t_1'-1} Q(k) \right\}$$

$$=^5 \sum_{i=1}^{i_0-1} \left\{ Q(t_{i-1}) \sum_{k=s_{i-1}}^{s_i-1} Q(k) + Q(s_i) \sum_{k=t_{i-1}}^{t_i-1} Q(k) \right\}$$

$$+ Q(t_{i_0-1}) \sum_{k=s_{i_0-1}}^{s_{i_0}-1} Q(k) + Q(s_{i_0}) \sum_{k=t_{i_0-1}}^{s_{i_0}-1} Q(k)$$

$$+ Q(s_{i_0}) \sum_{k=s_{i_0}}^{t_{i_0}-1} Q(k) + Q(t_{i_1}) \sum_{k=s_{i_0}}^{s_{i_0+1}-1} Q(k)$$

$$+ \sum_{i=i_0+2}^{m+1} \left\{ Q(s_{i-1}) \sum_{k=t_{i-2}}^{t_{i-1}-1} Q(k) + Q(t_{i-1}) \sum_{k=s_{i-1}}^{s_i-1} Q(k) \right\}$$

$$= \sum_{i=1}^{m} \left\{ Q(t_{i-1}) \sum_{k=s_{i-1}}^{s_i-1} Q(k) + Q(s_i) \sum_{k=t_{i-1}}^{t_i-1} Q(k) \right\}$$

$$= d(((0)_{s_1}\,,(1)_{t_1}\,,\ldots,(0)_{s_m-s_{m-1}}\,,(1)_{t_m-t_{m-1}})).$$

THEOREM 4. *Let* $((0)_{s_1}\,,(1)_{t_1}\,,\ldots,(0)_{s_m-s_{m-1}}\,,(1)_{t_m-t_{m-1}})$ *be an optimum schedule. Suppose*

(a) *this schedule maximizes* s_1 *in the class of optimum schedules, and*
(b) $s_i \geqslant t_i$ $(1 \leqslant i \leqslant m)$. *Then* $s_i = t_i$ $(1 \leqslant i \leqslant m)$.

PROOF: Assume on the contrary that $s_1 > t_1$.

[5] $Q = Q(0) = Q^{(1)}$.

CASE I. $s_1 \leqslant t_2$:

$$H(s_1, s_2; 0, t_1) < 0 \text{ (by (3.1) with } i = 1, \alpha = s_2, \beta = 0, \text{ and the maximality of } s_1). \tag{5.2}$$

$$\begin{aligned} &H(s_1, s_2; t_1, s_1) \leqslant 0 \text{ (by (3.2) with } i = 1, \alpha = s_1, \text{ and } \beta = s_1). \\ &H(t_1, s_1; 0, t_1) \geqslant 0 \text{ (by (3.2) with } i = 0, \alpha = t_1, \text{ and } \beta = t_1). \end{aligned} \tag{5.3}$$

Applying (2.6) to the rectangles determined by the system of points $(t_1, s_1, s_2; 0, t_1, s_1)$ we obtain $H(t_1, s_1; t_1, s_1) > 0$, in contradiction to (5.1).

CASE II. $s_1 > t_2$:

$$H(s_1, s_2; t_1, t_2) \geqslant 0 \text{ (by (3.2) with } i = 1, \alpha = s_1, \text{ and } \beta = t_2) \tag{5.4}$$

$$H(s_1, s_2; t_2, s_1) \geqslant 0 \text{ (by (3.7) with } i = 1, \gamma = s_1, \text{ and } \delta = s_1) \tag{5.5}$$

The inequalities (5.4)–(5.5) yield

$$H(s_1, s_2; t_1, s_1) = H(s_1, s_2; t_1, t_2) + H(s_1, s_2; t_2, s_1) \geqslant 0. \tag{5.6}$$

Finally using (5.2), (5.3), and (5.6) we apply (2.6) to the rectangles determined by the system of points $(t_1, s_1, s_2; 0, t_1, s_1)$ to conclude $H(t_1, s_1; t_1, s_1) > 0$, in contradiction to (5.1). Thus $s_1 = t_1$. We then write

$$\begin{aligned} d(((0)_{s_1}, (1)_{t_1}, \ldots, (0)_{s_m - s_{m-1}}, (1)_{t_m - t_{m-1}})) \\ = \sum_{k=0}^{s_1 - 1} Q(k) + Q(s_1) \sum_{k=0}^{s_1 - 1} Q(k) \\ + \sum_{i=1}^{m-1} \left\{ \bar{Q}(t'_{i-1}) \sum_{k=s'_{i-1}}^{s'_1 - 1} \bar{Q}(k) + \bar{Q}(s'_i) \sum_{k=t'_{i-1}}^{t'_1 - 1} \bar{Q}(k) \right\}, \end{aligned} \tag{5.7}$$

where

(a) $\bar{Q}(k) = Q(k + s_1)$, and

(b) $t'_i = t_{i+1} - s_1, s'_i = s_{i+1} - s_1 \qquad (0 \leqslant i < m)$.

It is clear that, apart from a scale factor, the final member (on the right-hand side) of (5.7) is of the form given in Lemma 1(ii) and hence we may reapply this argument to conclude that $s_i = t_i$ $(1 \leqslant i \leqslant m)$.

If we define

$$\mathscr{S} = \left\{ \begin{matrix} (i,j) : 0 \leqslant i \leqslant N, & 0 \leqslant j \leqslant N, & H(i, N; 0, i) < 0, \\ \text{and} & \min\limits_{\substack{0 \leqslant \alpha \leqslant i \\ 0 \leqslant \beta \leqslant i}} H(\alpha, i; 0, \beta) \geqslant 0 & \end{matrix} \right\},$$

then, by Theorem 4, $(s_1, t_1) \in \mathscr{S}$. Theorem 3 implies that $(s_1, t_1) = (i^*, i^*)$ where $i^* = \max\{i: (i, i) \in \mathscr{S}\}$.

6. Examples

We begin by assuming, without loss of generality, that $p^{(j)}(N^{(j)}) > 0$ $(j = 0, 1)$. Let

$$r^{(j)}(i) = \frac{p^{(j)}(i)}{Q^{(j)}(i-1)}, \qquad 1 \leqslant i < N^{(j)}, \qquad j = 0, 1.$$

Example 1. $r^{(0)}$ and $r^{(1)}$ non-decreasing:
We write

$$\begin{gathered} m(i,j) = Q^{(0)}(i)\, Q^{(1)}(j)[r^{(0)}(i+1) - r^{(1)}(j+1)], \\ 0 \leqslant i < N^{(0)}, \qquad 0 \leqslant j < N^{(1)}. \end{gathered} \tag{6.1}$$

Define

$$j_i^+ = \begin{cases} \min[N^{(1)}, 1 + \max\{j : m(i,j) \geqslant 0\}], & \text{if } \{j : m(i,j) \geqslant 0\} \neq \varnothing \\ 0, & \text{otherwise.} \end{cases} \tag{6.2}$$

Then (6.1) implies that $j_{i+1}^+ \geqslant j_i^+$.

Let $((0)_{s_1}, (1)_{t_1}, \ldots, (0)_{s_m - s_{m-1}}, (1)_{t_m - t_{m-1}})$ be an optimum schedule. If $((0)_{N^{(0)}}, (1)_{N^{(1)}})$ is not an optimum schedule then we may assume, by Lemmas 7 and 8, that $H(s_i, N^{(0)}; 0, t_i) < 0$ $(1 \leqslant i < m)$. We assert that there cannot exist a pair (s_i, t_i) which satisfy the conditions:

(a) $0 < s_i < N^{(0)}, \qquad 0 < t_i < N,$

(b) $t_i < j_{s_i}^+$.

If $i = 1$ then $H(s_i, s_{i+1}; 0, t_i) = 0$ by Lemma 4 and (6.2), which provides the contradiction. If $i > 1$ then, by Lemma 6 and (6.2), $H(s_i, N^{(0)}; 0, t_{i-1}) = 0$ and from (6.1) we can therefore conclude $H(s_i, N^{(0)}; 0, t_i) = 0$, a contradiction. A similar argument proves that there cannot exist a point of the form (s_i, t_i) (or (s_2, t_1) with $s_1 = 0$) such that

(a) $0 < s_i < N^{(0)}, \qquad 0 < t_i < N^{(1)}$,

(b) $t_i \geqslant j_{s_i}^+$,

(or $0 < s_2 < N^{(0)}, 0 < t_1 < N^{(1)}, t_1 \geqslant j_{s_2}^+$). It now follows that the schedule $((0)_a, (1)_{N^{(1)}}, (0)_{N^{(0)}-a})$ is optimum where

$$H(a, N^{(0)}; 0, N^{(1)}) = \min_{0 \leqslant \alpha \leqslant N^{(0)}} H(\alpha, N^{(0)}; 0, N^{(1)}).$$

EXAMPLE 2. $r^{(0)}$ and $r^{(1)}$ non-increasing:
Define

$$j_i^- = \begin{cases} \min[N^{(1)}, 1 + \max\{j : m(i,j) \leqslant 0\}], & \text{if } \{j : m(i,j) \leqslant 0\} \neq \varnothing, \\ 0, & \text{otherwise.} \end{cases} \tag{6.3}$$

Then (6.1) implies $j_{i+1}^- \geqslant j_i^-$. There exists a unique path from $(0, 0)$ to $(N^{(0)}, N^{(1)})$ which passes through the points $\{(i, j_i^-): 0 \leqslant i < N^{(0)}\}$ and for which $p_1(n) \leqslant j_{p_0(n)}^-$. This schedule is certainly optimal by (6.3).

EXAMPLE 3. $r^{(0)}$ non-increasing, $r^{(1)}$ non-decreasing:
With j_i^+ as defined by (6.2) we have $j_{i+1}^+ \leqslant j_i^+$. The analysis in Example 3 is similar to that in Example 1 and we may conclude that $((0)_a, (1)_{N^{(1)}}, (0)_{N^{(0)}-a})$ is an optimum schedule where a is a solution of

$$H(a, N^{(0)}; 0, N^{(1)}) = \min_{0 \leqslant \alpha \leqslant N^{(0)}} H(\alpha, N^{(0)}; 0, N^{(1)}).$$

EXAMPLE 4. $r^{(0)}$ non-decreasing, $r^{(1)}$ non-increasing:
With j_i^- as defined by (6.3) we have $j_{i+1}^- \leqslant j_i^-$. The analysis in Example 4 is similar to that in Example 1 and we may conclude that $((1)_b, (0)_{N^{(0)}}, (1)_{N^{(1)}-b})$ is an optimum schedule where b is a solution of

$$H(0, N^{(0)}; 0, b) = \min_{0 \leqslant \beta \leqslant N^{(1)}} H(0, N^{(0)}; 0, \beta).$$

EXAMPLE 5. Let

$$Q^{(j)}(i) = \begin{cases} 1 - i/N^{(j)}, & 0 \leqslant i \leqslant N^{(j)} \\ 0, & i > N^{(j)} \end{cases} \qquad (j = 0, 1).$$

We have

$$r^{(j)}(i) = 1/(N^{(j)} - i + 1) \qquad (1 \leqslant i \leqslant N^{(j)}, j = 0, 1),$$

so that by Example 1 the schedule $((0)_a, (1)_{N^{(1)}}, (0)_{N^{(0)}-a})$ is optimum where

$$H(a, N^{(0)}; 0, N^{(1)}) = \min_{0 \leqslant \alpha \leqslant N^{(0)}} H(\alpha, N^{(0)}; 0, N^{(1)}).$$

In this example

$$H(\alpha, N^{(0)}; 0, N^{(1)}) = \tfrac{1}{2}(N^{(0)} - \alpha)(N^{(1)} - N^{(0)} + \alpha).$$

If $N^{(1)} \geqslant N^{(0)}$ the minimum occurs at $\alpha = N^{(0)}$, while if $N^{(1)} < N^{(0)}$ the minimum occurs at $\alpha = N^{(1)}$.

The hypothesis that $N^{(j)} < \infty$ made in Section 1 is easily seen to be unnecessary for the analysis and may be replaced by the condition $E(T^{(j)}) < \infty$. In this connection we note

EXAMPLE 6. Let

$$p^{(j)}(i) = (1 - \theta_j)\, \theta_j^{i-1} \qquad (i = 1, 2,..., j = 0, 1)$$

with $0 < \theta_j < 1$. We note that $r^{(j)}(i) = 1/\theta_j - 1$ (independent of i). Thus if

$$E(T^{(0)}) = (1 - \theta_0)^{-1} \leqslant (1 - \theta_1)^{-1} = E(T^{(1)})$$

the optimum schedule is $((0)_\infty, (1)_\infty)$. Finally we note that if $\theta_0 = \theta_1 = \theta$ then all schedules are optimum, i.e., $d(\mathbf{x})$ is independent of $\mathbf{x}$.

EXAMPLE 7. Let

$$p^{(j)}(i) = \frac{\lambda_i^{i-1} e^{-\lambda i}}{(i-1)!} \qquad (i = 1, 2,..., j = 0, 1).$$

It is easy to show that

$$r^{(j)}(i + 1) \leqslant r^{(j)}(i) \qquad (i = 1, 2,..., j = 0, 1).$$

The determination of j_i^- requires the solution of a transcendental equation.

In the proceedings of a meeting which follows [1], Kendall gives a table of the distribution of lengths of computer programs run during a three month period on an IBM 7090. Kendall fits this to a Waring distribution. Let us recall that the Waring distribution with parameters x and a, $(x > a > 0)$ is given by

$$p(i) = (x - a)\frac{a(a+1)\cdots(a+i-2)}{x(x+1)\cdots(x+i-1)}, \qquad i = 1, 2,... .$$

One easily verifies that $r(i) = (x - a)/(x + i - 1)$ $(1 \leqslant i < \infty)$ and hence the optimal schedule can be determined by applying the results of Example 2.

We conclude this section by proving

THEOREM 5. *In the symmetric case, $Q^{(0)} = Q^{(1)} = Q$, the schedule $((0)_1, (1)_1, (0)_1, (1)_1, \ldots)$ is optimal if and only if $r(i) = 1 - Q(i)/Q(i-1)$ is non-increasing in i.*

PROOF (sufficiency):
We note that

$$m(i,j) = Q(i)\, Q(j)\{r(i+1) - r(j+1)\}$$

and hence

$$m(i,j) \quad \begin{matrix} \leqslant 0, & \text{if } j \leqslant i, \\ \geqslant 0, & \text{if } j \geqslant i. \end{matrix}$$

This proves $((0)_1, (1)_1, (0)_1, (1)_1, \ldots)$ is an optimum schedule.

PROOF necessity):
By Lemma 9 $m(i,j) \leqslant 0$ for $j \leqslant i$ and hence $r(i)$ is non-increasing in i.

7. FURTHER REMARKS

A. Let $\boldsymbol{\sigma} = (\sigma_0, \sigma_1)$ and $\boldsymbol{\tau} = (\tau_0, \tau_1)$ with $0 \leqslant \sigma_j \leqslant \tau_j \leqslant N^{(j)}$ $(j = 0, 1)$ and σ_j, τ_j integers. By *a path starting at* $\boldsymbol{\sigma}$ *and terminating at* $\boldsymbol{\tau}$ we shall mean a function $P : \mathbf{p} = (p_0, p_1)$ with domain $\{0, 1, \ldots, \tau_0 + \tau_1 - \sigma_0 - \sigma_1\}$ and range in R^2 which satisfies the conditions:

(i) $$\mathbf{p}(0) = \boldsymbol{\sigma},$$

(ii) $$\mathbf{p}(\tau_0 + \tau_1 - \sigma_0 - \sigma_1) = \boldsymbol{\tau},$$

(iii) $$\mathbf{p}(n) - p(n-1) \in \{(1,0), (0,1)\}, \quad 1 \leqslant n \leqslant \tau_0 + \tau_1 - \sigma_0 - \sigma_1 .$$

Let $\mathscr{P}(\boldsymbol{\sigma}, \boldsymbol{\tau})$ denote the family of all such paths. A point $\mathbf{v}$ is *a vertex* of the path $P \in \mathscr{P}(\boldsymbol{\sigma}, \boldsymbol{\tau})$ if any of the following conditions hold:

(i) $\mathbf{v} = \boldsymbol{\sigma}$,

(ii) $\mathbf{v} = \boldsymbol{\tau}$,

(iii) $\mathbf{v} = \mathbf{p}(n)$ with $1 \leqslant n < \tau_0 + \tau_1 - \sigma_0 - \sigma_1$ and $\mathbf{p}(n) - \mathbf{p}(n-1) \neq \mathbf{p}(n+1) - \mathbf{p}(n)$.

Let $\mathscr{V}(P)$ denote the set of vertices of the path P.

An integral over the path P is defined by

$$\int_P = \sum_{n=0}^{\tau_0+\tau_1-\sigma_0-\sigma_1} Q^{(0)}(p_0(n))\, Q^{(1)}(p_1(n)).$$

In Section 2 we proved that the search for an optimum processing schedule was equivalent to the problem of minimizing $\int_P$ over $P \in \mathscr{P}((0, 0), (N^{(0)}, N^{(1)}))$. We wish to indicate (without proof) certain relationships between the solutions of

$$\left\{P^* : \int_{P^*} = \min \left\{\int_P : P \in \mathscr{P}((0, 0), (N^{(0)}, N^{(1)}))\right\}\right\},$$

$$\left\{P^* : \int_{P^*} = \min \left\{\int_P : P \in \mathscr{P}(\boldsymbol{\sigma}, \boldsymbol{\tau})\right\}\right\}.$$

THEOREM 6. *Let*

$$P \sim ((0)_{s_1}, (1)_{t_1}, \ldots, (0)_{s_m - s_{m-1}}, (1)_{t_m - t_{m-1}}) \quad (P \in \mathscr{P}((0, 0), (N^{(0)}, N^{(1)}))$$

be an optimum schedule. Then there exists a path $\tilde{P} \in \mathscr{P}(\boldsymbol{\sigma}, \boldsymbol{\tau})$ *which minimizes* $\int_P$ *over* $\mathscr{P}(\boldsymbol{\sigma}, \boldsymbol{\tau})$ *such that*

$$\textit{range}\ \tilde{P} \supseteq \textit{range}\ P \cap \{(x, y) : \mathbf{v}_{\min} \leqslant (x, y) \leqslant \mathbf{v}_{\max}\}$$

where $\mathbf{v}_{\min}(\mathbf{v}_{\max})$ *is the minimum (resp. maximum) element of*

$$\mathscr{V}(P) \cap \{(x, y) : \boldsymbol{\sigma} \leqslant (x, y) \leqslant \boldsymbol{\tau}\}$$

provided that this latter set is non-empty.

B. We sketch here an alternate method for finding an optimum schedule. The method has certain computational advantages but yields no information about the structure of an optimum schedule.

Define

$$f(\boldsymbol{\sigma}) = \min \left\{\int_P : P \in \mathscr{P}(\boldsymbol{\sigma}, (N^{(0)}, N^{(1)}))\right\}.$$

It is clear that if $f(\boldsymbol{\sigma})$ is known for all $\boldsymbol{\sigma}$ then optimum schedules can easily be obtained from the following recursion:

$$\mathbf{p}(n+1) = \begin{cases} \mathbf{p}(n) + (1, 0), & \text{if } f(\mathbf{p}(n) + (1, 0)) \leqslant f(\mathbf{p}(n) + (0, 1)), \\ \mathbf{p}(n) + (0, 1), & \text{if } f(\mathbf{p}(n) + (0, 1)) \leqslant f(\mathbf{p}(n) + (1, 0)), \end{cases}$$

where either choice is possible if equality holds. On the other hand it is clear that f itself satisfies the following recursion:

$$f(i,j) = Q^{(0)}(i)\, Q^{(1)}(j) + \min(f(i+1,j), f(i,j+1)), \tag{7.1}$$

with the boundary conditions $f(N^{(0)},j) = f(i, N^{(1)}) = 0$. Thus (7.1) enables us to effectively compute $f(i,j)$ for all (i,j), proceeding one row and one column at a time.

C. If the distributions of the service times $\{T^{(j)}\}$ are continuous, then the problem of finding an optimum schedule may still be formulated, although the optimum schedules may require infinitely many switchings in a finite time interval and thus be physically meaningless. As before we define

$$Q^{(j)}(x) = \Pr\{\omega: T^{(j)}(\omega) > x\} \qquad (j = 0, 1)$$

and assume $E(T^{(j)}) < \infty$. We define a schedule by means of a non-decreasing function $s(t)$ satisfying:

(i) $$s(0) = 0,$$

(ii) $$0 \leqslant s'(t) \leqslant 1 \quad \text{(a.e.)}.$$

The quantity to be minimized is

$$d(s) = \int_0^\infty Q^{(0)}(t - s(t))\, Q^{(1)}(s(t))\, dt.$$

(This formula is derived by the method of proof of Lemma 1.)

By an appropriate approximation technique the methods developed in Sections 2–5 can be used to characterize optimum schedules for this problem. The details are tedious albeit straightforward and hence we omit them, and content ourselves with an example.

EXAMPLE. Suppose

(i) $$\frac{d}{dx} F^{(i)}(x) \geqslant 0, \text{ where } F^{(i)}(x) = \frac{1}{Q^{(i)}(x)} \frac{d}{dx} Q^{(i)}(x),$$

(ii) $$F^{(i)}(x) > F^{(i)}(0), \qquad \text{for} \quad x > 0,$$

(iii) $$F^{(0)}(0) = F^{(1)}(0).$$

Then if $s(t)$ is the value of s where $Q^{(0)}(t-s)\, Q^{(1)}(s)$ assumes its minimum value on $0 \leqslant s \leqslant t$, then $s(t)$ is an optimum schedule.

The optimality of s is trivial. Assumptions (i)–(iii) are imposed only to show that s is well defined and is indeed a schedule. This follows in a straightfoward manner from the assumptions.

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